

BLOWUP AND FIXED POINTS

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ABSTRACT. Blowing up a point p in a manifold M builds a new manifold \widehat{M} in which p is replaced by the projectivization of the tangent space $T_p M$. This well-known operation also applies to fixed points of diffeomorphisms, yielding continuous homomorphisms between automorphism groups of M and \widehat{M} . The construction for maps involves a loss of regularity and is not unique at the lowest order of differentiability. Fixed point sets and other aspects of blowup dynamics at the singular locus are described in terms of derivative data; C^0 data are not sufficient to determine much about these issues.

Topological generalizations of the blowup construction prove to be much less natural than the classical versions, and no lifting homomorphism for homeomorphism groups can be constructed.

1. INTRODUCTION

The construction for blowing up points and subspaces which is a mainstay in algebraic geometry, especially in the resolution of singularities, is investigated here from a dynamical point of view. The blowup in this sense of a point p in a smooth manifold M^n is a map of manifolds, $q: V^n \rightarrow M^n$, which is a homeomorphism away from $q^{-1}(\{p\})$, and for which $q^{-1}(\{p\})$ is a nonempty compact set, classically a projective space. Real and complex versions of the construction are both considered here. Some other notions of blowing up points appear in dynamics, notably in studies of normal forms for vector fields and in constructions which delete a fixed point set and manipulate an open cylinder.

The principal theme presented here is that the algebraic geometers' form of blowup (Section 3) is so natural that it easily induces continuous homomorphisms on diffeomorphism groups (Sections 4 and 6), defined by an explicit model given in Section 4. These lifting homomorphisms render derivative data at the space level since the exceptional locus in blowup is a projectivized tangent space. For complex manifolds and biholomorphic maps this rendering works nicely, as it does for real C^∞ manifolds and diffeomorphisms. Dynamical consequences of the construction are laid out in Section 7.

However, for finitely differentiable diffeomorphisms the loss of regularity (Example 6.2) in blowup becomes interesting and leads to a second theme: at the lowest order of differentiability we find that a C^1 diffeomorphism fixing a point might lift to many homeomorphisms with variant dynamics and quotient projections (Section 8). The argument for this nonuniqueness claim uses local C^0 conjugacy facts for hyperbolic fixed points of diffeomorphisms and suggests that the dynamical universality of the classical blowup is much more distinctive than the spatial or single-map aspects of the construction. This C^0 variation of blowup also indicates limitations on neighborhood-based invariants of dynamics. The paper's third theme is that while the blowup notion is easy to generalize in a C^0 context (Section 2), greatly enlarging on the topological effects of classical blowup (Section 5), when

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we give up differentiability entirely it turns out that the only reasonable generalized blowups which allow every homeomorphism of the base manifold to lift are necessarily homeomorphic to the base manifold (Theorem 8.4).

Topologists are familiar with blowup as a construction tool and stabilizing device in four-manifold topology. Nash [7] posed questions, since amplified by others, on the equivalence relation on manifolds which blowup generates. Major progress on Nash's space-level question was made in [1, 2], and especially in the work of Mikhalkin [6]. Blowup equivalence of diffeomorphisms or group actions may be ripe for study after those advances.

We close the introduction with some notational conventions. B^n denotes the open ball $\{\mathbf{x} \in \mathbf{R}^n : |\mathbf{x}| < 1\}$, while the closed disk $D^n = \{\mathbf{x} \in \mathbf{R}^n : |\mathbf{x}| \leq 1\}$. The derivative of h at p is written $Dh|_p$. The projection from a Cartesian product onto its j -th factor is denoted pr_j .

Mapping spaces for pairs appear frequently below. If M is a manifold and $A \subset M$ then $\text{Aut}_{C^k}(M, A)$ denotes the space of C^k diffeomorphisms f of M such that $f(A) = A$; f is not obliged to fix A pointwise. The analogous reading is used for spaces of homeomorphisms or other self-maps such as $\text{Homeo}(M, A)$.

2. BLOWUPS OF A MANIFOLD OR MAP

This section considers nonclassical, merely continuous versions of the notion of blowing up a manifold or a map between manifolds. Although these are easy to construct topologically, they do not ordinarily have the universality properties of the classical constructions, and it seems likely that homomorphisms such as those exhibited in Theorem 6.1 are a distinguishing feature of the classical constructions whose description begins in Section 3.

Definition 2.1. A *topological blowup* of an n -manifold M^n at a point $p \in M$ is a quotient map $q: V^n \rightarrow M^n$ such that

- (1) V^n is also an n -manifold,
- (2) $\Sigma := q^{-1}(\{p\})$ is a connected, compact, nonempty subset of V , and
- (3) $q|_{V \setminus \Sigma}: V \setminus \Sigma \rightarrow M \setminus \{p\}$ is a homeomorphism.

A topological blowup of a self-map $f: M^n \rightarrow M^n$ of an n -manifold M^n at a fixed point $p = f(p) \in M$ of f is a topological blowup $q: V^n \rightarrow M^n$ of M at p together with a self-map $\tilde{f}: V \rightarrow V$ such that this diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\tilde{f}} & V \\ \downarrow q & & \downarrow q \\ M & \xrightarrow[f]{} & M, \end{array}$$

i.e., $q \circ \tilde{f} = f \circ q: V \rightarrow M$.

$\Sigma = q^{-1}(\{p\})$ is called the *exceptional locus* and M is sometimes described as a “blowdown” of V . Our first examples are constructed top-downwards, by beginning with V and Σ .

One could very reasonably add to Definition 2.1 the requirement that $V \setminus \Sigma$ should be dense in V . We shall not do so in this paper, but note in advance the relevance of this density condition in Theorem 8.4.

A subset Σ of a manifold V^n is *cellular* if there are closed sets $S_i \subset V^n$ such that $S_1 \supseteq S_2 \supseteq \cdots \supseteq S_i \supseteq S_{i+1} \cdots$, $\Sigma = \bigcap_1^\infty S_i$ and for every i , $S_i \cong D^n$ is a disk imbedded with bicollared boundary.

Example 2.2. If Σ is a cellular subset of V^n then $M = V/\Sigma$ is a manifold and the quotient map $q: V \rightarrow V/\Sigma$ is a blowup of M at the image of Σ .

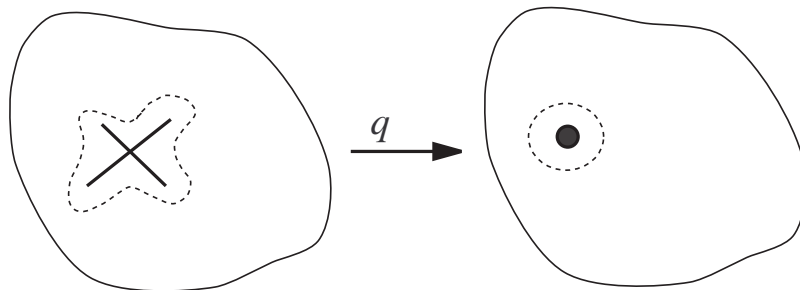


FIGURE 1. Blowup with a cellular exceptional locus.

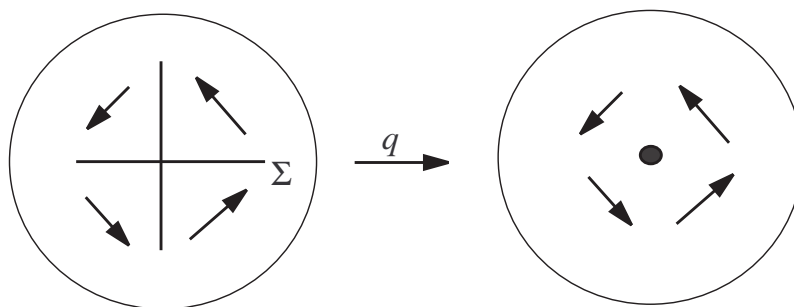


FIGURE 2. Blowing up a rotation.

See Figure 1 for a sketch of this sort of example, in which Σ is a finite polyhedral tree.

Example 2.3. If $g: V^n \rightarrow V^n$ is a map which preserves a cellular subset $\Sigma \subset V$, then g descends to a map $f: V/\Sigma \rightarrow V/\Sigma$ and g together with the quotient map $q: V \rightarrow V/\Sigma$ defines a blowup of f .

Examples 2.2 and 2.3 are misleading, inasmuch as V^n and M^n are homeomorphic. This is not usually the case, and the replacement of $\{p\}$ by Σ can affect the global topology of a manifold in drastic ways.

Example 2.4. If $V^n \cong W^n \# X^n$ is a connected sum and $\Sigma = X^{(n-1)}$ is the codimension-one skeleton of a CW structure for X which has one top-dimensional cell, then the quotient map $q: V \rightarrow V/\Sigma \cong W$ is a blowup of W with exceptional locus Σ .

For instance, if V^n is a compact, connected manifold and Σ is the codimension-one skeleton of a cell structure for V which has one top cell then $V/\Sigma \cong S^n$.

Example 2.5. Examples of topological blowups for maps as well as spaces are not hard to produce, and one is sketched in Figure 2. Suppose that $g: V^n \rightarrow V^n$ is periodic of period r (so $g^r = \text{Id}_V$), that g has a fixed point a , and that the action of the cyclic group C_r generated by g is effective on V and locally linear at a . Form a g -invariant polyhedral tree Σ with r legs emanating from a , beginning with a short segment J based at a so that $J \setminus \{a\}$ lies in the open dense set of V on which C_r acts freely. If J is sufficiently short and becomes smooth in a linear model for the action near a , then $\Sigma = \bigcup_1^r g^i(J)$ is the desired tree, lying in a Euclidean ball about a . The periodic map g descends to a periodic map f on V/Σ and the pair $q: V \rightarrow V/\Sigma$, $g: V \rightarrow V$ defines a blowup of f .

Remark 2.6. The distinctive property of the exceptional locus Σ in a blowup $q: V^n \rightarrow M^n$ arises from the requirement that $V^n \setminus \Sigma \cong M^n \setminus \{p\}$ and concerns deleted neighborhoods: There is an open neighborhood U of Σ in V such that $U \setminus \Sigma \cong B^n \setminus \{\mathbf{0}\}$.

Such a neighborhood provides collared codimension-one spheres exhibiting a connected sum structure for V , so Example 2.4 is more typical than it might appear, although up to this point we have allowed non-CW compacta to appear as exceptional loci. The classical blowup construction of algebraic geometry exploits an instance of this neighborhood structure in projective space, exactly along the lines of Example 2.4.

3. THE CLASSICAL MODEL CONSTRUCTION

The most classical form of blowing up is performed at the origin in \mathbf{F}^n , where \mathbf{F} is \mathbf{R} or \mathbf{C} . Our account mostly follows [5], and a good description of the construction and the properties which extend it from the affine model to other varieties is found in [3].

$\mathbf{P}(\mathbf{F}^n)$ denotes the projective space of the vector space \mathbf{F}^n , defined as the quotient $\mathbf{P}(\mathbf{F}^n) = (\mathbf{F}^n \setminus \{\mathbf{0}\}) / \sim$, where $\mathbf{v} \sim \mathbf{w}$ if and only if there exists $\lambda \in \mathbf{F} \setminus \{0\}$ such that $\mathbf{v} = \lambda \mathbf{w}$. Square brackets denote homogeneous coordinates on a projective space, so that the image in $\mathbf{P}(\mathbf{F}^n)$ of $(v_1, \dots, v_n) \in \mathbf{F}^n \setminus \{\mathbf{0}\}$ is written $[v_1, \dots, v_n]$. We will also use $[\mathbf{v}]$ to label the image in $\mathbf{P}(\mathbf{F}^n)$ of a nonzero vector \mathbf{v} in \mathbf{F}^n .

Let $X \subset \mathbf{F}^n \times \mathbf{P}(\mathbf{F}^n)$ be the subset

$$X = \{((x_1, \dots, x_n), [y_1, \dots, y_n]) : \text{for every } j, k, x_j y_k = x_k y_j\}$$

and let

$$\begin{aligned} q: X &\rightarrow \mathbf{F}^n \\ (\mathbf{x}, [\mathbf{y}]) &\mapsto \mathbf{x} \end{aligned}$$

be the restriction of first-coordinate projection $\text{pr}_1: \mathbf{F}^n \times \mathbf{P}(\mathbf{F}^n) \rightarrow \mathbf{F}^n$.

Lemma 3.1. $X = \{(\mathbf{x}, [\mathbf{y}]) : \text{there exists } \mu \in \mathbf{F} \text{ such that } \mathbf{x} = \mu \mathbf{y}\}$. In addition, X is a subvariety of $\mathbf{F}^n \times \mathbf{P}(\mathbf{F}^n)$, q is an algebraic map, and preimages under q ,

$$q^{-1}(\{\mathbf{x}\}) = \begin{cases} \{(\mathbf{x}, [\mathbf{x}])\}, & \text{if } \mathbf{x} \neq \mathbf{0}, \\ \{(\mathbf{0}, [\mathbf{y}]) : [\mathbf{y}] \in \mathbf{P}(\mathbf{F}^n)\}, & \text{if } \mathbf{x} = \mathbf{0}, \end{cases}$$

are such that q is an isomorphism away from the origin and the fiber of q over the origin is isomorphic to the projective space $\mathbf{P}(\mathbf{F}^n)$. \square

$\Sigma = q^{-1}(\{\mathbf{0}\}) \cong \mathbf{P}(\mathbf{F}^n)$ is usually called the *exceptional locus* or *exceptional divisor*. The quotient map q is sometimes called the *blowdown* map, since it alters X only by identifying Σ to a point (thus “blowing down Σ ”).

First-coordinate projection in $\mathbf{F}^n \times \mathbf{P}(\mathbf{F}^n)$ defined the blowdown map q , and second-coordinate projection determines the structure of a neighborhood of the exceptional locus in classical blowups.

Lemma 3.2. *Second-coordinate projection restricts to X as*

$$\begin{aligned} (\mathbf{x}, [\mathbf{x}]) &\mapsto [\mathbf{x}] \\ (\mathbf{0}, [\mathbf{y}]) &\mapsto [\mathbf{y}] \end{aligned}$$

and identifies X with the universal line bundle over $\mathbf{P}(\mathbf{F}^n)$, i.e., the \mathbf{F}^1 -bundle over this projective space whose fiber at $[\mathbf{y}]$ is the line $\{\lambda \mathbf{y} : \lambda \in \mathbf{F}\}$ through the origin and \mathbf{y} in \mathbf{F}^n . Σ is identified with the zero section in this bundle, so the normal bundle of Σ in X is identified with the universal line bundle. \square

We will return to this bundle structure in Section 5.

4. NATURALITY PROPERTIES AND MANIFOLD CONSTRUCTIONS

Any self-map of $(\mathbf{F}^n, \{\mathbf{0}\})$ such that h is differentiable at the origin and $Dh|_0$ is a \mathbf{F} -linear isomorphism lifts to a map \widehat{h} of X , where \widehat{h} is defined by:

$$(4.1) \quad \begin{aligned} \widehat{h}: (\mathbf{x}, [\mathbf{x}]) &\mapsto (h(\mathbf{x}), [h(\mathbf{x})]), \\ \widehat{h}: (\mathbf{0}, [\mathbf{y}]) &\mapsto (\mathbf{0}, [Dh|_0(\mathbf{y})]). \end{aligned}$$

Lemma 4.1. *If $h: (\mathbf{F}^n, \{\mathbf{0}\}) \rightarrow (\mathbf{F}^n, \{\mathbf{0}\})$ is a continuous map which is differentiable at $\mathbf{0}$, and if $Dh|_0$ is a \mathbf{F} -linear isomorphism, then the map $\widehat{h}: (X, \Sigma) \rightarrow (X, \Sigma)$ defined above is continuous and makes this diagram commute*

$$\begin{array}{ccc} (X, \Sigma) & \xrightarrow{\widehat{h}} & (X, \Sigma) \\ q \downarrow & & \downarrow q \\ (\mathbf{F}^n, \{\mathbf{0}\}) & \xrightarrow{h} & (\mathbf{F}^n, \{\mathbf{0}\}), \end{array}$$

i.e., $q \circ \widehat{h} = h \circ q: (X, \Sigma) \rightarrow (\mathbf{F}^n, \{\mathbf{0}\})$.

Proof. The claim that $q \circ \widehat{h} = h \circ q$ follows immediately from $q = \text{pr}_1|_X$ and $\widehat{h}(\Sigma) = \Sigma$. Because $q: X \setminus \Sigma \rightarrow \mathbf{F}^n \setminus \{\mathbf{0}\}$ is a homeomorphism and h is continuous, the continuity claim only needs to be confirmed at points of Σ .

The restriction of \widehat{h} to $\Sigma = \{\mathbf{0}\} \times \mathbf{P}(\mathbf{F}^n)$ is the projectivization $\mathbf{P}(Dh|_0)$ of a linear isomorphism, so this restriction is a C^∞ diffeomorphism on Σ . Because $h(\mathbf{x}) = Dh|_0(\mathbf{x}) + R(\mathbf{x})$, where $R(\mathbf{x}) = o(|\mathbf{x}|)$ as $|\mathbf{x}| \rightarrow 0$, \widehat{h} is continuous on the normal line $\{(\lambda\mathbf{x}, [\lambda\mathbf{x}])\}$ through $(\mathbf{0}, [\mathbf{x}]) \in \Sigma$:

$$\begin{aligned} \widehat{h}(\lambda\mathbf{x}, [\lambda\mathbf{x}]) &= (h(\lambda\mathbf{x}), [h(\lambda\mathbf{x})]) \\ &= (h(\lambda\mathbf{x}), [Dh|_0(\lambda\mathbf{x}) + R(\lambda\mathbf{x})]) \\ &= (h(\lambda\mathbf{x}), [(\lambda|\mathbf{x}|)^{-1}Dh|_0(\lambda\mathbf{x}) + (\lambda|\mathbf{x}|)^{-1}R(\lambda\mathbf{x})]) \\ &= (h(\lambda\mathbf{x}), [Dh|_0(|\mathbf{x}|^{-1}\mathbf{x}) + (\lambda|\mathbf{x}|)^{-1}R(\lambda\mathbf{x})]), \end{aligned}$$

which tends to $(\mathbf{0}, [Dh|_0(\mathbf{x})])$ as $\lambda \rightarrow 0$, with convergence uniform in $|\mathbf{x}|^{-1}\mathbf{x}$. Therefore, by the triangle inequality, \widehat{h} is continuous at every point of Σ . \square

The next lemma follows from the Chain Rule. Recall that a map or homeomorphism of pairs $(X, \Sigma) \rightarrow (X, \Sigma)$ is required to carry Σ to itself but need not restrict to the identity on Σ .

Lemma 4.2. *Let $g, h: (\mathbf{F}^n, \{\mathbf{0}\}) \rightarrow (\mathbf{F}^n, \{\mathbf{0}\})$ be continuous maps which are differentiable at $\mathbf{0}$ and have \mathbf{F} -linear isomorphisms as their derivatives at the origin. Then $\widehat{g \circ h} = \widehat{g} \circ \widehat{h}: (X, \Sigma) \rightarrow (X, \Sigma)$.* \square

Since $\widehat{\text{Id}_{\mathbf{F}^n}} = \text{Id}_X$, the Lemma shows that $h \mapsto \widehat{h}$ defines a homomorphism

$$\beta: \text{Aut}_{C^1, \mathbf{F}}(\mathbf{F}^n, \{\mathbf{0}\}) \rightarrow \text{Homeo}(X, \Sigma),$$

where the \mathbf{F} decoration indicates that the derivatives are required to be \mathbf{F} -linear.

Proposition 4.3. *A smooth real or complex manifold M can be blown up at any point p to produce a quotient map from a smooth real or complex manifold \widehat{M} ,*

$$q: (\widehat{M}, \Sigma) \rightarrow (M, \{p\}),$$

which restricts to an isomorphism $q|: \widehat{M} \setminus \Sigma \xrightarrow{\cong} M \setminus \{p\}$. If M is modelled on \mathbf{F}^n then $\Sigma \cong \mathbf{P}(\mathbf{F}^n)$.

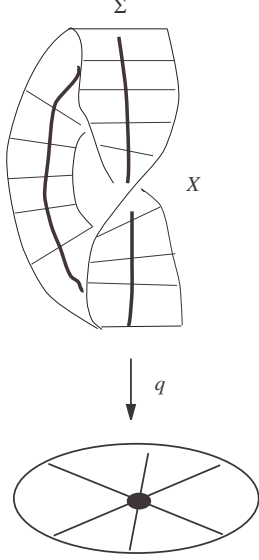


FIGURE 3. Normal structure of the exceptional locus.

Proof. Lemma 4.2 shows that origin-preserving coordinate changes with \mathbf{F} -linear derivatives act as automorphisms of $q: (X, \Sigma) \rightarrow (\mathbf{F}^n, \{\mathbf{0}\})$. If $\phi_i: U \rightarrow \mathbf{F}^n$ are local coordinate systems ($i = 1, 2$) on a neighborhood U of p in M then $(\phi_1 \circ \phi_2^{-1}): X \rightarrow X$ gives a change of coordinates on the model blowup. The homomorphism properties established in Lemma 4.2 show that blowup coordinate change maps satisfy the cocycle condition, yielding a consistent pasting construction for \bar{M} from the data defining M . \square

The same naturality properties used above give diffeomorphism blowups on smooth manifolds, which are treated in detail in Theorem 6.1.

5. TOPOLOGY

This section describes the effects of the classical blowup construction on topology. We begin with the model construction at the origin in \mathbf{F}^n , where the space X and the normal bundle of Σ in X are identified with the universal line bundle over the projective space Σ .

This bundle description gives a picture of the blowdown map which may be helpful (see Figure 3). Let $S(\mathbf{F}^n)$ denote the unit sphere in \mathbf{F}^n ; then a tubular neighborhood of Σ in X is identified with the mapping cylinder of the Hopf map $h: S(\mathbf{F}^n) \rightarrow \mathbf{P}(\mathbf{F}^n)$ and the blowdown quotient on this tubular neighborhood is the natural map between the mapping cylinders for this Hopf map and for the constant map $c: S(\mathbf{F}^n) \rightarrow \text{point}$, i.e., the map of pairs $(\text{MapCyl}(h), \mathbf{P}(\mathbf{F}^n)) \rightarrow (\text{MapCyl}(c), \{\text{point}\})$.

In the complex case a bit of attention is required to the line bundles playing roles in this discussion. McDuff and Salamon [5] describe these identifications or computations carefully:

- (a) $\nu_X(\Sigma)$ is identified with the universal line bundle L over the projective space $\Sigma \cong \mathbf{P}(\mathbf{C}^n)$;
- (b) the first Chern class $c_1(\nu_X(\Sigma)) = -c$, where c is the positive or canonical generator of $H^2(\Sigma; \mathbf{Z})$;
- (c) the normal line bundle to the hyperplane section in $\mathbf{P}(\mathbf{C}^{n+1})$ has first Chern class $c_1(\nu_{\mathbf{P}(\mathbf{C}^{n+1})} \mathbf{P}(\mathbf{C}^n)) = c$; and

(d) the normal line bundle to the hyperplane section in the conjugate complex structure $\overline{\mathbf{P}(\mathbf{C}^{n+1})}$ has first Chern class $c_1\left(\nu_{\overline{\mathbf{P}(\mathbf{C}^{n+1})}}\overline{\mathbf{P}(\mathbf{C}^n)}\right) = -c$.

The real blowup of a point in a Riemann surface has $\Sigma \cong \mathbf{R}P^1 \cong S^1$, where the model space X is the nonorientable line bundle over S^1 whose total space is a Möbius band. Thus, for surfaces the mapping cylinder description of blowing up and down suggests that blowing up a point has the global topological effect of sewing in a crosscap. This is true, and in general the global effect of blowing up a point is a connected sum operation, as in Example 2.4 and Remark 2.6. For real blowups,

$$\widehat{M}^n \cong M^n \# \mathbf{R}P^n,$$

and for complex blowups

$$\widehat{M}^n \cong M^n \# \overline{\mathbf{C}P^n}.$$

A conjugate complex structure appears in the second connected sum because of the determinations of line bundles in the preceding paragraph.

6. BLOWING UP MAPS AT A FIXED POINT

The naturality properties of the classical blowup construction suggest that (4.1) shows how to extend blowup to a homomorphism of diffeomorphism groups. This is possible, but a kink develops in the C^k case.

The regularity loss in the theorem below is formally due to a division when one considers the homogeneous coordinate side of the formula for \widehat{h} . More geometrically, the blowup construction renders tangential data for h as spatial data for \widehat{h} since Σ is the space of lines in $T_p M$, so the loss of one derivative should be expected. An example is worked out below to show that the the loss of a derivative is genuine.

Theorem 6.1. *Classical blowup of a point $p \in M$ determines continuous, injective homomorphisms*

$$\beta: \text{Aut}_{C^k}(M^n, \{p\}) \rightarrow \text{Aut}_{C^{k-1}}(\widehat{M}, \Sigma)$$

(in the real case) and

$$\beta: \text{Aut}_{\text{Holo}}(M^n, \{p\}) \rightarrow \text{Aut}_{\text{Holo}}(\widehat{M}, \Sigma)$$

(in the complex analytic case), where $\beta(h) = \widehat{h}$ in both cases.

Proof. Away from Σ we know that \widehat{h} and h may be identified, so the regularity issue only arises along Σ . Along Σ we have defined \widehat{h} to be the projectivization of the linear map $Dh|_{\mathbf{0}}$, so \widehat{h} is infinitely differentiable in those directions.

If for \mathbf{x} near $\mathbf{0}$ we have $h(\mathbf{x}) = \sum_{j=1}^k D^j h|_{\mathbf{0}}(\mathbf{x}, \dots, \mathbf{x}) + R(\mathbf{x})$, where $R(\mathbf{x}) = o(|\mathbf{x}|^k)$ as $\mathbf{x} \rightarrow \mathbf{0}$, then

$$\begin{aligned} \widehat{h}(\mathbf{x}, [\mathbf{x}]) &= \left(\sum_{j=1}^k D^j h|_{\mathbf{0}}(\mathbf{x}, \dots, \mathbf{x}) + R(\mathbf{x}), \left[\sum_{j=1}^k D^j h|_{\mathbf{0}}(\mathbf{x}, \dots, \mathbf{x}) + R(\mathbf{x}) \right] \right) \\ &= \left(\sum_{j=1}^k D^j h|_{\mathbf{0}}(\mathbf{x}, \dots, \mathbf{x}) + R(\mathbf{x}), \left[\sum_{j=1}^k D^j h|_{\mathbf{0}}(\mathbf{x}/|\mathbf{x}|, \mathbf{x}, \dots, \mathbf{x}) + R(\mathbf{x})/|\mathbf{x}| \right] \right). \end{aligned}$$

The division by $|\mathbf{x}|$ inside the homogeneous coordinates gives a zero-th order term of $Dh|_{\mathbf{0}}(\mathbf{x}/|\mathbf{x}|)$, similarly reduces the degree of the other homogeneous terms in the Taylor expansion, and reduces by one the order of vanishing for the remainder term. The resulting expansion of \widehat{h} near $(0, [\mathbf{x}]) \in \Sigma$ shows that we lose one partial derivative of \widehat{h} along the fibers of the normal bundle $\nu_X(\Sigma)$, compared to the degree of smoothness of h at $\mathbf{0}$. (Recall from Section 4 that \widehat{h} is C^∞ along Σ .)

The partial derivatives of \widehat{h} along the singular locus and normal to it are continuous, through order $k - 1$, so \widehat{h} is C^{k-1} at points of Σ by the familiar theorem deducing (Fréchet) differentiability from continuous partial derivatives.

Lemma 4.2 and surrounding discussion show that β is a homomorphism.

Once we know that \widehat{h} is continuous, it follows that β is injective, since h determines \widehat{h} on the dense subset $\widehat{M} \setminus \Sigma$.

β is continuous because the C^r distance between diffeomorphisms on M^n majorizes the C^{r-1} distance between their blowups on $\widehat{M} \setminus \Sigma$ and the C^{r-1} distance between those blowups on Σ . \square

Example 6.2. This is a two-dimensional example of the regularity loss from C^1 to C^0 indicated in the theorem.

Let $g: \mathbf{R} \rightarrow \mathbf{R}$ be defined by $g(x) = x + x|x|$, so that $g'(x) = 1 + 2|x|$. g is a C^1 diffeomorphism, but not C^2 , and $g(0) = 0$.

Define $h: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ to be the C^1 diffeomorphism $h(x, y) = (g(x), y)$. This map preserves the origin and blows up there to $\widehat{h}: ((x, y), [x, y]) \mapsto ((x + x|x|, y), [x + x|x|, y])$. The parametrized line $t \mapsto (t, mt)$ of slope $m \neq 0$ in the plane is covered in the blownup plane by the C^∞ parametric curve $c: t \mapsto ((t, mt), [t, mt])$, and the composite

$$\begin{aligned} \widehat{h} \circ c: t &\mapsto ((t + t|t|, mt), [t + t|t|, mt]) \\ &= ((t + t|t|, mt), [m^{-1} + m^{-1}|t|, 1]) \end{aligned}$$

is continuous but not differentiable at $t = 0$, since in the usual local coordinate system about $[0, 1] \in \mathbf{P}(\mathbf{R}^2)$ the second component becomes $t \mapsto m^{-1} + m^{-1}|t|$: therefore \widehat{h} is not differentiable, and not even Gâteaux differentiable, at $c(0) = ((0, 0), [1, m]) \in \Sigma$.

Similar examples for C^k to C^{k-1} regularity loss are available for all $k \geq 1$.

Theorem 8.4 indicates that C^0 to C^0 lifting of automorphisms through blowups is problematic for other reasons.

7. DYNAMICS

The dynamics of a classically blown-up diffeomorphism \widehat{h} off, on, and near the singular locus are described in terms of basic features of the original diffeomorphism h .

If $h: (M, \{p\}) \rightarrow (M, \{p\})$ is a diffeomorphism fixing p then the restriction of q gives

$$\widehat{h}|_{\widehat{M} \setminus \Sigma} \cong h|_{M \setminus \{p\}},$$

so blowup does not modify dynamics far from the exceptional locus.

Lemma 7.1. *On the exceptional locus $\Sigma = \mathbf{P}(T_p M)$*

$$\widehat{h}|_\Sigma \cong \mathbf{P}(Dh|_p)$$

is a projectivized linear map, with the new fixed point set given by

$$(7.1) \quad \Sigma \cap \text{Fix}(\widehat{h}) = \coprod_{\lambda \in \mathbf{F}} \mathbf{P}(E_\lambda),$$

where $E_\lambda = \ker(\lambda I - Dh|_p)$ and \coprod denotes a disjoint union.

Proof. Equation 4.1 defines $\widehat{h}: (\mathbf{0}, [\mathbf{y}]) \mapsto (\mathbf{0}, [Dh|_{\mathbf{0}}(\mathbf{y})])$ in the model case, so the restriction of \widehat{h} to Σ is the projectivization of the derivative $Dh|_p$.

Therefore, fixed points of \widehat{h} in $\Sigma = \mathbf{P}(T_p M)$ are solutions of $[Dh|_p(v)] = [v]$, i.e. projective equivalence classes of tangent vectors v for which there exist scalars $\lambda \in \mathbf{F}$ satisfying $Dh|_p(v) = \lambda v$, that is, projective equivalence classes of eigenvectors

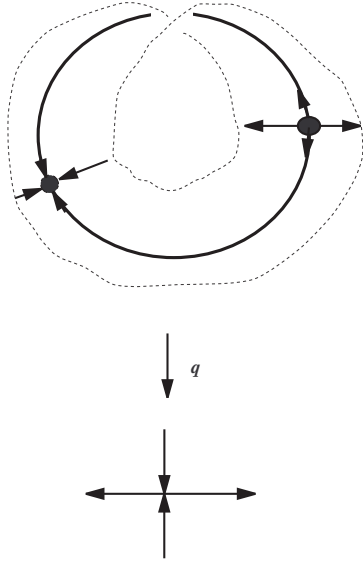


FIGURE 4. Blowing up a hyperbolic fixed point.

of $Dh|_p$. Equation 7.1 describes the set of all such projective classes as a disjoint union of projectivized subspaces of $T_p M$. \square

Derivative computations for \hat{h} at points of Σ and in directions not tangent to Σ will involve the loss of order noted in Theorem 6.1. These are omitted here – see the displayed equation in the proof of that theorem for the appearance of second derivatives of h in the first derivative of \hat{h} . Despite this derivative complication, a qualitative picture of part of the dynamics of \hat{h} normal to Σ is provided by some naturality observations.

First, if N^k is a C^1 submanifold of N^n passing through p and \widehat{M} is the blowup of M^n at p then there is a C^0 submanifold V^k of \widehat{M} such that V is homeomorphic to \widehat{N} and $q_N: (\widehat{N}, \Sigma_N) \rightarrow (N, \{p\})$ is equivalent to $q_M|_V: (V, V \cap \Sigma_M) \rightarrow (N, \{p\})$. The main step in checking this claim is a confirmation that the closure of $q_M^{-1}(N \setminus \{p\})$ in \widehat{M} meets Σ_M in $\mathbf{P}(T_p N^k)$.

Second, if $h: (M^n, N^k, \{p\}) \rightarrow (M^n, N^k, \{p\})$ is a C^1 diffeomorphism then \hat{h} preserves the submanifold V^k defined in the preceding paragraph.

In particular, if p is a hyperbolic fixed point of h then this applies to the stable and unstable manifolds at p , and also to any invariant local submanifolds tangent to other invariant subspaces of $T_p M$, such as eigenspaces of $Dh|_p$. Because $\Sigma \cap V = \mathbf{P}(T_p N)$, in Figure 4 the blownup one-dimensional stable submanifold at p meets Σ in the point corresponding to the appropriate eigenspace of $Dh|_p$, while the blownup unstable manifold meets Σ in the point corresponding to another one-dimensional eigenspace.

This low-dimensional example suggests the behavior of blowups at hyperbolic fixed points, but is a bit simpler than the general case, which we sketch now.

Suppose that p is a hyperbolic fixed point of h , that E_s and E_u are the stable and unstable subspaces of $T_p M$, and that the stable and unstable submanifolds at p are W_s and W_u . W_s and W_u blow up at p to give invariant submanifolds V_s , V_u which meet Σ in a submanifold (either $\mathbf{P}(E_s)$ or $\mathbf{P}(E_u)$) which is invariant and forward attracting (respectively backward attracting). The dynamics of \hat{h} restricted to $\mathbf{P}(E_s)$ or $\mathbf{P}(E_u)$ are those of a projectivized linear map.

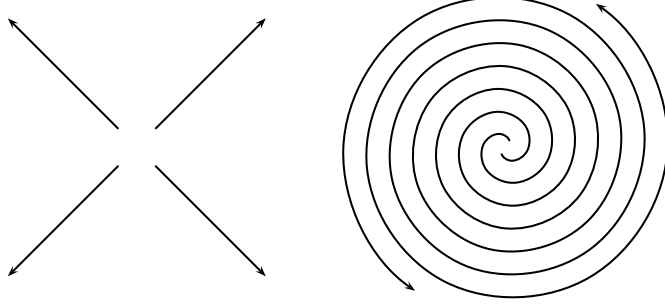


FIGURE 5. Topologically conjugate hyperbolic fixed points.

8. VARIANT BLOWUPS

C^0 data for h are not enough to determine $\widehat{h}|_\Sigma$ as a homeomorphism. For example, in the hyperbolic case we can apply local conjugacy results to obtain lots of homeomorphisms blowing up a given C^1 diffeomorphism. A handy reference for these facts on topological conjugacy is [4, Sec. 6.3], which gives a proof of this result on local equivalence of hyperbolic fixed points:

Remark 8.1. Topological conjugacy classes of hyperbolic diffeomorphisms with p as an isolated fixed point are determined by the dimensions and orientations of the stable and unstable manifolds of these diffeomorphisms at p .

For example, if M is even-dimensional over \mathbf{R} , p is a fixed point for h_i ($i = 1, 2$), and $Dh_1|_p$ is diagonalizable with all eigenvalues lying in the interval $\lambda > 1$, while $Dh_2|_p$ has only non-real eigenvalues, all satisfying $|\lambda| > 1$, then h_1 and h_2 are locally conjugate near p . Figure 5 suggests how different in appearance such topologically conjugate diffeomorphisms can be, and indicates that the cause of the phenomenon is a familiar difficulty: we can unwind a spiral with a continuous automorphism, but not with a smooth one.

A global topological conjugacy $\phi: M \rightarrow M$ from h_1 to h_0 leads to a variant blowup of h_0 with \widehat{h}_1 as the covering homeomorphism and $\phi \circ q$ as blowdown map.

$$\begin{array}{ccc}
 \widehat{M} & \xrightarrow{\widehat{h}_1} & \widehat{M} \\
 q \downarrow & & \downarrow q \\
 M & \xrightarrow{h_1} & M \\
 \phi \downarrow & & \downarrow \phi \\
 M & \xrightarrow{h_0} & M
 \end{array}$$

The dynamics on Σ of these conjugacy-induced blowups can differ dramatically from those of the classical construction. Fixed point sets on Σ may differ drastically in dimension as we run over diffeomorphisms h_1 which are topologically conjugate to h_0 , ranging from empty to discrete to connected and high-dimensional. We emphasize this variability with a proposition.

Proposition 8.2. *Let $h: M^n \rightarrow M^n$ be a diffeomorphism with an isolated hyperbolic fixed point p . If the stable and unstable subspaces $E_p^s, E_p^u \subseteq T_p M$ are both even-dimensional then there are conjugacy-induced topological blowups \widehat{h} of h such that $\Sigma \cap \text{Fix}(\widehat{h})$ is of any of these sorts:*

- (a) empty,
- (b) discrete, containing any even number of points between 2 and n , or
- (c) positive-dimensional, with any dimension between 1 and $-1 + \max(\dim(E_p^u), \dim(E_p^s))$.

Proof. In each case a C^0 conjugacy as described in Remark 8.1 between h and another diffeomorphism g with a hyperbolic fixed point at p yields the topological blowup. Our job here is to allocate eigenvalues for $Dg|_p$ and apply Lemma 7.1 to \widehat{g} .

If every eigenvalue $\lambda \in \mathbf{C} \setminus \mathbf{R}$ then $Dg|_p$ has no real eigenvectors and the classical blowup's $\Sigma \cap \text{Fix}(\widehat{g})$ is empty.

If $Dg|_p$ has k distinct real eigenvalues and $n-k$ complex eigenvalues appearing in conjugate pairs, then k must be even but may otherwise assume any value between 0 and n . In this case we see k isolated fixed points for \widehat{g} on Σ .

Positive-dimensional fixed point sets arise from repeated real eigenvalues for $Dg|_p$. These may appear in combinations so that the multiplicities of real eigenvalues form partitions of some even numbers $0 \leq e_u \leq \dim(E_p^u)$, $0 \leq e_s \leq \dim(E_p^s)$:

$$\begin{aligned} k_1 + k_2 + \cdots + k_q &= e_u, \\ l_1 + l_2 + \cdots + l_r &= e_s, \end{aligned}$$

where k_i, l_j are the multiplicities of unstable, respectively stable, real eigenvalues of $Dg|_p$. Each of these real eigenvalues produces a component of the fixed point set $\Sigma \cap \text{Fix}(\widehat{g})$ which is diffeomorphic to a projective space: If λ is a real eigenvalue of multiplicity m then $\mathbf{P}(E_\lambda) \cong \mathbf{P}(\mathbf{R}^m) \cong \mathbf{R}P^{m-1}$. The largest dimension arising in this way $\max(\dim(\mathbf{P}(E_p^u)), \dim(\mathbf{P}(E_p^s)))$. \square

Note that $\Sigma \cap \text{Fix}(\widehat{g})$ might have components of different dimensions. The complex case is similar, but the fixed point set must be nonempty.

The next few results indicate that topological blowups are necessarily limited in naturality.

Lemma 8.3. *Let N^k be a connected topological manifold without boundary. If $\{x_i\}$ and $\{y_i\}$ are sequences in $N \times (0, \infty)$ such that $\{\text{pr}_2(x_i)\} \rightarrow \infty$, $\{\text{pr}_2(y_i)\} \rightarrow \infty$, and both of these sequences in \mathbf{R} are strictly increasing, then there is a homeomorphism $g: N \times (0, \infty) \rightarrow N \times (0, \infty)$ such that for every i , $g(x_i) = y_i$, and such that for some $\varepsilon > 0$ the restriction $g|_{(0, \varepsilon]}$ is the identity.*

Proof. This is a consequence of the following version of the homogeneity of manifolds: for any $x, y \in N$ there exists an isotopy from Id_N to a homeomorphism which carries x to y .

In more detail, g can be built in segments which are pasted together. We may apply a homeomorphism of the form $\rho: (x, t) \mapsto (x, \psi(t))$, where $\psi: (0, \infty) \rightarrow (0, \infty)$ is a homeomorphism, to arrange that $\text{pr}_2(\rho(x_i)) = \text{pr}_2(y_i)$ for all i . Continue the argument with the sequence $\{\rho(x_i)\}$ replacing $\{x_i\}$.

Let $0 < s_1 < \text{pr}_2(x_1)$ and let $\phi_t: N \rightarrow N$ be an isotopy over $s_1 \leq t \leq \text{pr}_2(x_1)$ such that $\phi_{s_1} = \text{Id}_N$ and $\phi_{\text{pr}_2(x_1)}(\text{pr}_1(x_1)) = \text{pr}_1(y_1)$. Define the first two pieces of $g: N \times (0, \infty) \rightarrow N \times (0, \infty)$ by $g|_{N \times (0, s_1]} = \text{Id}_{N \times (0, s_1]}$ and $g|_{N \times [s_1, \text{pr}_2(x_1)]}: (x, t) \mapsto (\phi_t(x), t)$. Subsequent segments are defined by taking an isotopy ϕ_t over $\text{pr}(x_{i-1}) \leq t \leq \text{pr}_2(x_i)$ which starts with the $\phi_{\text{pr}(x_{i-1})}$ already selected and ends at a homeomorphism which carries $\phi_{\text{pr}(x_{i-1})}(x_i)$ to y_i . \square

The topological blowups $q: (V, \Sigma) \rightarrow (M, \{p\})$ of greatest interest will share with the classical construction the property that $V \setminus \Sigma$ is dense in V . This line of argument shows that in no such case can we find a lifting construction for homeomorphisms.

Theorem 8.4. *Let $q: (V^n, \Sigma) \rightarrow (M^n, \{p\})$ be a topological blowup of the manifold M^n at p such that at least two points lie on the frontier of Σ in V . If $n \geq 2$ then there is a homeomorphism $h: (M, \{p\}) \rightarrow (M, \{p\})$ which does not lift through q .*

Proof. Suppose that $x, y \in \Sigma$ are distinct points and that $\{x_i\}, \{y_i\}$ are sequences in $V \setminus \Sigma$ such that $x = \lim_{i \rightarrow \infty} x_i$ and $y = \lim_{i \rightarrow \infty} y_i$. We may assume that both sequences lie in a neighborhood U of Σ which admits a homeomorphism $f: U \setminus \Sigma \xrightarrow{\cong} S^{n-1} \times (0, \infty)$ and that the sequences $\{\text{pr}_2 \circ f(x_i)\}$ and $\{\text{pr}_2 \circ f(y_i)\}$ are strictly increasing and converge to ∞ .

Form a third sequence $\{z_i\}$ such that each z_{2j} is one of the y_i , each z_{2j+1} is one of the x_i , and the real sequence $\{\text{pr}_2 \circ f(z_i)\}$ is strictly increasing and converges to ∞ . $\{z_i\}$ is not convergent in V , but all three of the sequences $\{q(x_i)\}$, $\{q(y_i)\}$, and $\{q(z_i)\}$ converge to p in M .

Since $n \geq 2$, Lemma 8.3 implies that there is a homeomorphism $h: (M, \{p\}) \rightarrow (M, \{p\})$ such that for every i , $h(q(x_i)) = q(z_i)$. If h is covered by $\tilde{h}: V \rightarrow V$ then $\{\tilde{h}(x_i)\}$ is divergent although $\{x_i\}$ converges to x , so \tilde{h} is not continuous. \square

Corollary 8.5. *Suppose that $n \geq 2$ and let $q: (V^n, \Sigma) \rightarrow (M^n, \{p\})$ be a topological blowup of the manifold M^n at p such that $V \setminus \Sigma$ is dense in V . If every homeomorphism of $(M, \{p\})$ is covered by a homeomorphism of (V, Σ) , then $\Sigma = \{pt.\}$ and q is a homeomorphism.* \square

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